Discrete Mathematics 121 (1993) 223–228
North-Holland

Hamiltonian paths and hamiltonian connectivity in graphs

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Received 8 November 1990
Revised 10 May 1991

Abstract

Wei, B., Hamiltonian paths and hamiltonian connectivity in graphs, Discrete Mathematics 121 (1993) 223–228.

Let G be a 2-connected graph with n vertices such that $d(u)+d(v)+d(w)-|N(u)\cap N(v)\cap N(w)|\geqslant n+1$ holds for any triple of independent vertices u, v and w. Then for any distinct vertices u and v such that $\{u,v\}$ is not a cut vertex set of G, there is a hamiltonian path between u and v. In particular, if G is 3-connected, then G is hamiltonian-connected. This is closely related to the main result in Flandrin et al. (1991) and generalizes a theorem of Ore (1963) and a theorem of Faudree et al. (1989).

1. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. $V(G)$, or just V, and $E(G)$, or just E, denote the set of vertices and the set of edges of a graph G, respectively. For an integer $k\geqslant 1$, we denote

$$\sigma_k=\min\left\{\sum_{i=1}^{k} d(v_i)\colon \{v_1,\ldots,v_k\}\subseteq V \text{ is an independent set of } G\right\}$$

and

$$\bar{\sigma}_3=\min\left\{\sum_{i=1}^{3} d(v_i)-|N(v_1)\cap N(v_2)\cap N(v_3)|\colon \{v_1,v_2,v_3\}\subseteq V \text{ is an independent set of } G\right\}.$$

A graph G is called hamiltonian-connected (Hc for short) if there exists a hamiltonian path (H-path for short) between any two distinct vertices of G.

Recently, Flandrin et al. [2] proved the following result.

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Theorem 1.1. *Let G be a 2-connected graph with n vertices such that* $\bar{\sigma}_3 \geqslant n$. *Then G is hamiltonian.*

Our main purpose is to prove the following theorem.

Theorem 1.2. *Let G be a 2-connected graph with n vertices such that* $\bar{\sigma}_3 \geqslant n+1$. *Then for any* $u \neq v$ *and* $\{u, v\}$ *is not a vertex cut set of G, there exists a H-path between u and v in G. In particular, if G is 3-connected, then G is Hc.*

Theorem 1.2 generalizes the following results.

Theorem 1.3 (Ore [3]). *Let G be a 2-connected graph with n vertices such that* $\sigma_2 \geqslant n+1$. *Then G is Hc.*

Actually, by $\sigma_2 \geqslant n+1$, we can easily get that G is 3-connected.

Theorem 1.4 (Faudree et al. [1]). *Let G be a 3-connected graph with n vertices. If, for every pair of nonadjacent vertices u and v, we have* $|N(u) \cup N(v)| > 2n/3$, *then G is Hc.*

The complete bipartite graph $K_{n,n}$ ($n \geqslant 3$) shows that the lower bound $n+1$ on $\bar{\sigma}_3$ in Theorem 1.2 is best possible.

2. Several lemmas

In order to prove Theorem 1.2, we shall first prove several lemmas and give some notations. $P=P(x, y)$: $y_1 \ldots y_j$ denotes a path between vertices x and y ($y_1=x$, $y_j=y$) of length $j-1$ with orientation from x to y. If $u \neq y$, u^+ denotes the successor of u on P, and if $u \neq x$, u^- denotes its predecessor of u on P. If $S \subseteq V(P)=\{y_1, \ldots, y_j\}$, then $S^+=\{u^+ : u \in S-\{y\}\}$ and $S^-=\{u^- : u \in S-\{x\}\}$. If $H, S \subseteq V$, then $N_H(S)$ denotes a set of vertices of H which are adjacent to S. In particular, when $H=V$, $S=\{u\}$, we let $N_{V(G)}(u)=N(u)$ and $d(u)=d_G(u)=|N(u)|$. If R is a component of $G-P$, we label $N_P(V(R))=\{x_1, \ldots, x_s\}$ in the order from x to y. We denote $\bar{e}(\{u, v, w\}; P)= d_P(u)+d_P(v)+d_P(w)-|N_P(u) \cap N_P(v) \cap N_P(w)|$ for a path P, where $\{u, v, w\} \subseteq V$ and $d_P(z)=|N_P(z)|$, $z \in \{u, v, w\}$. For a subpath P_i of P, let $N^+_{P_i}(u)$ ($N^-_{P_i}(u)$) denote $(N_{P_i}(u))^+$ ($(N_{P_i}(u))^-$).

Lemma 2.1. *Let* P: $y_1 \ldots, y_j$ ($y_1=x$, $y_j=y$) *be a path between x and y in G. If there is an independent set* $\{u, v, w\}$ *such that*

$$(N^+_{P_i}(u) \cup N^+_{P_i}(v)) \cap (N_{P_i}(w) \cup (N_{P_i}(u) \cap N_{P_i}(v))) = \emptyset,$$

$$|N^+_{P_i}(u) \cup N^+_{P_i}(v)| \geqslant |N_{P_i}(u) \cup N_{P_i}(v)| - \theta$$

for some subpath $P_i \subseteq P$ *and* $\theta \geqslant 0$, *then* $\bar{e}(\{u, v, w\}; P_i) \leqslant |P_i| + \theta$.

Proof. Since $(N^+_{P_i}(u)\cup N^+_{P_i}(v))\cap(N_{P_i}(w)\cup(N_{P_i}(u)\cap N_{P_i}(v)))=\emptyset$ and $|N^+_{P_i}(u)\cup N^+_{P_i}(v)|\geqslant |N_{P_i}(u)\cup N_{P_i}(v)|-\theta$, we have

$$\begin{aligned}d_{P_i}(w)&\leqslant|P_i|-|N^+_{P_i}(u)\cup N^+_{P_i}(v)|-|N_{P_i}(u)\cap N_{P_i}(v)-N_{P_i}(w)|\\&\leqslant|P_i|-|N_{P_i}(u)\cup N_{P_i}(v)|+\theta-|N_{P_i}(u)\cap N_{P_i}(v)-N_{P_i}(w)|\\&=|P_i|-d_{P_i}(u)-d_{P_i}(v)+|N_{P_i}(u)\cap N_{P_i}(v)|-|N_{P_i}(u)\cap N_{P_i}(v)-N_{P_i}(w)|+\theta.\end{aligned}$$

Thus,

$$d_{P_i}(u)+d_{P_i}(v)+d_{P_i}(w)-|N_{P_i}(u)\cap N_{P_i}(v)\cap N_{P_i}(w)|\leqslant|P_i|+\theta. \qquad \square$$

Clearly, Lemma 2.1 remains valid if $N^+_{P_i}(u)$ and $N^+_{P_i}(v)$ are replaced, respectively, by $N^-_{P_i}(u)$ and $N^-_{P_i}(v)$.

Lemma 2.2. *Let G satisfy the conditions of Theorem 1.2 and P: $y_1\cdots y_j$ be a longest path between x and y ($y_1=x$, $y_j=y$). If $\{x, y\}$ is not a cut set of G and $j\leqslant n-1$, then:*

(i) $N^-_P(R)\cap N^+_P(R)=\emptyset$.

(ii) *For all $x_i\in N_P(R)$, we have*

$$N_P(x_i^+)\cap N_P(R)-\{x_i, y_1, y_j\}=N_P(x_i^-)\cap N_P(R)-\{x_i, y_1, y_j\}=\emptyset.$$

Proof. (i) By contradiction, choose $x_i^+\in N^-_P(R)\cap N^+_P(R)$. Since $\{x, y\}$ is not a vertex cut set of G, $|N^+_P(R)|\geqslant 2$. Thus, $\{x_i^+, x_t^+, v\}$ is an independent set of G for any $i\neq t$ and any $v\in V(R)$. Without loss of generality, let $t>i$. Denote $P_1: y_1\cdots x_i^+$, $P_2: x_{i+1}\cdots y_j$; then $(N^+_{P_1}(x_i^+)\cup N^+_{P_1}(v))\cap(N^+_{P_1}(x_{i+1}^+)\cup(N_{P_1}(x_i^+)\cap N_{P_1}(v)))=\emptyset$ (otherwise G contains a path from x to y which is longer than P, a contradiction) and because $x_i^+\notin N_{P_1}(x_i^+)\cup N_{P_1}(v)$, we have $|N^+_{P_1}(x_i^+)\cup N^+_{P_1}(v)|=|N_{P_1}(x_i^+)\cup N_{P_1}(v)|$. Thus, by Lemma 2.1, we can get $\bar{e}(\{x_i^+, x_{i+1}^+, v\}; P_1)\leqslant|P_1|$.

Note that $|N^+_{P_2}(x_i^+)\cup N^+_{P_2}(v)|\geqslant|N_{P_2}(x_i^+)\cup N_{P_2}(v)|-1$. For the same reason, $\bar{e}(\{x_i^+, x_{i+1}^+, v\}; P_2)\leqslant|P_2|+1$. Therefore, we have $\bar{e}(x_i^+, x_{i+1}^+, v\}; P)\leqslant|P|+1$. On the other hand, we have $d_{G-P}(x_i^+)+d_{G-P}(x_{i+1}^+)+d_{G-P}(v)\leqslant n-|P|-1$. Hence, $d(x_i^+)+d(x_{i+1}^+)+d(v)\leqslant n+|N(x_i^+)\cap N(x_{i+1}^+)\cap N(v)|$, a contradiction.

(ii) By contradiction, choose $x_t\in N_P(x_i^+)\cap N_P(R)-\{x_i, y_1, y_j\}$ for some $1<t<j$. Then $\{x_t^-, x_t^+, v\}$ is an independent set of G for any $v\in V(R)$. We consider the following two cases.

Case 1: $1<t<i$. Denote P_1: $y_1\cdots x_t^-$, P_2: $x_t^+\cdots y_j$. Then we have $(N^+_{P_1}(x_t^-)\cup N^+_{P_1}(v))\cap(N_{P_1}(x_t^+)\cup(N_{P_1}(x_t^-)\cap N_{P_1}(v)))=\emptyset$ and $|N^+_{P_1}(x_t^-)\cup N^+_{P_1}(v)|=|N_{P_1}(x_t^-)\cup N_{P_1}(v)|$. Thus, by Lemma 2.1, we can get $\bar{e}(\{x_t^-, x_t^+, v\}; P_1)\leqslant|P_1|$ and $\bar{e}(\{x_t^-, x_t^+, v\}; P_2)\leqslant|P_2|$ (for the same reason). Thus, $\bar{e}(\{x_t^-, x_t^+, v\}; P)\leqslant|P|+1$. On the other hand, we have $d_{G-P}(x_t^-)+d_{G-P}(x_t^+)+d_{G-P}(v)\leqslant n-|P|-1$. Hence, we can get a contradiction as in the proof of (i).

Case 2: $j>t>i$. Then $\{x_t^-, x_t^+, v\}$ is an independent set. Denote $P_1: y_1\cdots x_t^-$, P_2: $x_t^+\cdots y_j$ again. Using the same method as in case 1, we can also get a contradiction.

Therefore, $N_P(x_i^+)\cap N_P(R)-\{x_i, y_1, y_j\}=\emptyset$. Symmetrically, we can prove $N_P(x_i^-)\cap N_P(R)-\{x_i, y_1, y_j\}=\emptyset$. $\square$

Lemma 2.3. *Let G satisfy the conditions of Theorem 1.2, and P: $y_1 \cdots y_j$ ($y_1 = x$, $y_j = y$) be a longest path between x and y in G. If $j \leqslant n-1$, then $x_i^- x_i^{++} \notin E$ and $x_i^{--} x_i^+ \notin E$ hold for any $x_i \in N_P(R)$.*

Proof. By contradiction, say $x_i^- x_i^{++} \in E$.

If $1 \leqslant i \leqslant s-1$, then $\{x_i^+, x_{i+1}^-, v\}$ is an independent set for any $v \in V(R)$. Denote P_1: $y_1 \cdots x_i^-$, P_2: $x_i^+ \cdots y_j$. Since

$$(N_{P_1}^+(x_{i+1}^-) \cup N_{P_1}^+(v)) \cap (N_{P_1}(x_i^+) \cup (N_{P_1}(x_{i+1}^-) \cap N_{P_1}(v))) = \emptyset,$$

$$(N_{P_2}^-(x_i^+) \cup N_{P_2}^-(v)) \cap (N_{P_2}(x_{i+1}^-) \cup (N_{P_2}(x_i^+) \cap N_{P_2}(v))) = \emptyset$$

and

$$|N_{P_1}^+(x_{i+1}^-) \cup N_{P_1}^+(v)| = |N_{P_1}(x_{i+1}^-) \cup N_{P_1}(v)|,$$

$$|N_{P_2}^-(x_i^+) \cup N_{P_2}^-(v)| = |N_{P_2}(x_i^+) \cup N_{P_2}(v)|,$$

by Lemma 2.1, we can get $\bar{e}(\{x_i^+, x_{i+1}^-, v\}; P) \leqslant |P| + 1$, which leads to a contradiction.

If $i = s$, then $\{x_s^+, x_{s-1}^-, v\}$ is an independent set for any $v \in V(R)$. Denote P_1: $y_1 \cdots x_s^-$, P_2: $x_s^+ \cdots y_j$. Then

$$|N_{P_1}^+(x_{s-1}^-) \cup N_{P_1}^+(v))| = |N_{P_1}(x_{s-1}^-) \cup N_{P_1}(v)|,$$

$$|N_{P_2}^-(x_s^+) \cup N_{P_2}^-(v)| = |N_{P_2}(x_s^+) \cup N_{P_2}(v)|$$

and

$$(N_{P_1}^+(x_{s-1}^-) \cup N_{P_1}^+(v)) \cap (N_{P_1}(x_s^+) \cup (N_{P_1}(x_{s-1}^-) \cap N_{P_1}(v))) = \emptyset,$$

$$(N_{P_2}^-(x_s^+) \cup N_{P_2}^-(v)) \cap (N_{P_2}(x_{s-1}^-) \cup (N_{P_2}(x_s^+) \cap N_{P_2}(v))) = \emptyset.$$

Thus, by Lemma 2.1, $\bar{e}(\{x_{s-1}^-, x_s^+, v\}; P) \leqslant |P|$, which leads to a contradiction.

Symmetrically, we can also prove that $x_i^+ x_i^{--} \notin E$. $\square$

Lemma 2.4. *Let G satisfy the conditions of Theorem 1.2, and P: $y_1 \cdots y_j$ ($y_1 = x$, $y_j = y$) be a longest path between x and y in G. If $j \leqslant n-1$, then $\{y_1, y_j\} \subseteq N_P(R)$.*

Proof. By contradiction, say $y_1 \notin N_P(R)$, we consider the following two cases.

Case 1: $x_1^- x_1^+ \notin E$.

Case 1.1: $x_2^- x_2^+ \notin E$. Then $\{x_1^-, x_2^-, v\}$ is an independent set for any $v \in V(R)$. Denote P_1: $y_1 \cdots x_1^-$, P_2: $x_1^+ \cdots x_2^{--}$, P_3: $x_2^- \cdots y_j$. Then we have $d_{P_t}(x_1^-) + d_{P_t}(x_2^-) \leqslant |P_t|$ and $d_{P_t}(v) = 0$ ($t = 1, 2$). Since $x_2^- x_2^+ \notin E$, by Lemma 2.2(ii), $(N_{P_3}^-(x_2^-) \cup N_{P_3}^-(v)) \cap (N_{P_3}(x_1^-) \cup (N_{P_3}(x_2^-) \cap N_{P_3}(v))) = \emptyset$ and $|N_{P_3}^-(x_2^-) \cup N_{P_3}^-(v)| = |N_{P_3}(x_2^-) \cup N_{P_3}(v)|$. Thus, by Lemma 2.1, we have $\bar{e}(\{x_1^-, x_2^-, v\}; P_3) \leqslant |P_3|$. Hence, $\bar{e}(\{x_1^-, x_2^-, v\}; P) \leqslant |P|$, which leads to a contradiction.

Case 1.2: $x_2^- x_2^+ \in E$. By Lemma 2.3, $x_2^- x_2^{++} \notin E$, $x_2^+ x_2^{--} \notin E$.

Case 1.2.1: $y_j \notin N_P(R)$. Then $\{x_1^+, x_2^{++}, v\}$ is an independent set for any $v \in V(R)$. Denote P_1: $y_1 \cdots x_1^+$, P_2: $x_1^{++} \cdots x_2^-$, P_3: $x_2^{++} \cdots y_j$. By Lemma 2.2(ii),

$(N^+_{P_3}(x_1^+)\cup N^+_{P_3}(v))\cap(N_{P_3}(x_2^{++})\cup(N_{P_3}(x_1^+)\cap N_{P_3}(v)))=\emptyset$. Since $x_1^-x_1^+\notin E$, $x_2^-x_2^{++}\notin E$ and $|N^+_{P_3}(x_1^+)\cup N^+_{P_3}(v)|=|N_{P_3}(x_1^+)\cup N_{P_3}(v)|-1$, by Lemma 2.1, we can get $\bar{e}(\{x_1^+, x_2^{++}, v\}; P_t)\leqslant|P_t|$ $(t=1, 2, 3)$. Therefore, $\bar{e}(\{x_1^+, x_2^{++}, v\}; P)\leqslant|P|+1$, which leads to a contradiction.

Case 1.2.2: $y_j\in N_P(R)$. As we have shown in case 1.2.1, y_j must be in $N^+_P(x_1^+)\cap N_P(x_1^+)\cap N_P(v)$ and $\{x_1^-, y_j^-, v\}$ is an independent set for any $v\in V(R)$. Denote $P_1: y_1\cdots x_1^-$, $P_2: x_1^+\cdots y_j$. Since $x_1^-x_1^+\notin E$, by Lemma 2.2(ii), $(N^-_{P_2}(x_1^-)\cup N^-_{P_2}(v))\cap(N_{P_2}(y_j^-)\cup(N_{P_2}(x_1^-)\cap N_{P_2}(v)))=\emptyset$ and $|N^-_{P_2}(x_1^-)\cup N^-_{P_2}(v)|=|N_{P_2}(x_1^-)\cup N_{P_2}(v)|$. Using the same method as before, we can get $\bar{e}(\{x_1^-, y_j^-, v\}; P)\leqslant|P|+1$, which leads to a contradiction.

Case 2: $x_1^-x_1^+\in E$. By Lemma 2.3, $x_1^-x_1^{++}\notin E$, $x_1^{--}x_1^+\notin E$ (if $x_1^-\neq y_1$).

Case 2.1: $x_2^-x_2^+\notin E$.

If $x_1^-=y_1$, then we choose $\{x_1^-, x_2^-, v\}$ for some $v\in V(R)$ and denote $P_1: y_1\cdots x_2^{--}$, $P_2: x_2^-\cdots y_j$. Because $x_2^-x_2^+\notin E$, by Lemma 2.2(ii), we have $(N^-_{P_2}(x_2^-)\cup N^-_{P_2}(v))\cap(N_{P_2}(x_1^-)\cup(N_{P_2}(x_2^-)\cap N_{P_2}(v)))=\emptyset$ and $|N^-_{P_2}(x_2^-)\cup N^-_{P_2}(v)|=|N_{P_2}(x_2^-)\cup N_{P_2}(v)|$. Using the same method as before, we can get $\bar{e}(\{x_1^-, x_2^-, v\}; P)\leqslant|P|+1$, which leads to a contradiction.

If $x_1^-\neq y_1$, then $x_1^{--}x_1^+\notin E$. Choose $\{x_1^{--}, x_2^-, v\}$ for some $v\in V(R)$ and denote $P_1: y_1\cdots x_1^{--}$, $P_2: x_1^-\cdots x_2^{--}$, $P_3: x_2^-\cdots y_j$. Since $x_2^-x_2^+\notin E$, we have $(N^-_{P_3}(x_2^-)\cup N^-_{P_3}(v))\cap(N_{P_3}(x_1^{--})\cup(N_{P_3}(x_2^-)\cap N_{P_3}(v)))=\emptyset$ and $|N^-_{P_3}(x_2^-)\cup N^-_{P_3}(v)|=|N_{P_3}(x_2^-)\cup N_{P_3}(v)|$. Using the same method as before, we can get a contradiction.

Case 2.2: $x_2^-x_2^+\in E$. By Lemma 2.3, $x_2^{--}x_2^+\notin E$, $x_2^{++}x_2^-\notin E$. Choose $\{x_2^{++}, x_1^+, v\}$ for some $v\in V(R)$ and denote $P_1: y_1\cdots x_1^+$, $P_2: x_1^{++}\cdots x_2^-$, P_3: $x_2\cdots y_j$. Using the same method as before, we can get $\bar{e}(\{x_1^+, x_2^{++}, v\}; P_1)\leqslant|P_1|+1$, $\bar{e}(\{x_1^+, x_2^{++}, v\}; P_2)\leqslant|P_2|$. Therefore, by $d_{G-p}(x_1^+)+d_{G-p}(x_2^{++})+d_{G-P}(v)\leqslant n-|P|-1$, we can get $\bar{e}(\{x_1^+, x_2^{++}, v\}; P_3)\geqslant|P_3|+1$. Thus, $x_2x_2^{++}\in E$ or $y_j\in N_P(v)\cap N_P(x_1^+)\cap N^+_P(x_1^+)$.

Case 2.2.1: $x_2x_2^{++}\in E$. Then $\{x_1^-, x_2^+, v\}$ is an independent set for any $v\in V(R)$. Denote $P_1: y_1\cdots x_1^-$, $P_2: x_1\cdots x_2$, $P_3: x_2^+\cdots y_j$. Then we have $\bar{e}(\{x_1^-, x_2^+, v\}; P_1)\leqslant|P_1|$, $\bar{e}(\{x_1^-, x_2^+, v\}; P_2)\leqslant|P_2|$ (because $x_2^{--}\notin N_P(v)\cup N_P(x_1^-)\cup N_P(x_2^+)$) and $(N^-_{P_3}(x_2^+)\cup N^-_{P_3}(v))\cap(N_{P_3}(x_1^-)\cup(N_{P_3}(x_2^+)\cap(N_{P_3}(v)))=\emptyset$ by Lemma 2.2 and $|N^-_{P_3}(x_2^+)\cup N^-_{P_3}(v)|=|N_{P_3}(x_2^+)\cup N_{P_3}(v)|$. Thus, by Lemma 2.1, we have $\bar{e}(\{x_1^-, x_2^+, v\}; P_3)\leqslant|P_3|$. Using the same method as before, we can get a contradiction.

Case 2.2.2: $y_j\in N_P(v)\cap N_P(x_1^+)\cap N^+_P(x_1^+)$. In this subcase, we choose $\{x_1^-, y_j^-, v\}$ for some $v\in V(R)$ and denote $P_1: y_1\cdots x_1^-$, P_2: $x_1\cdots x_2$ and $P_3: x_2^+\cdots y_j$. For the same reason as that in case 2.2.1, we can also get a contradiction.

Therefore, Lemma 2.4 is true. □

3. The proof of Theorem 1.2

By contradiction, say $P: y_1\cdots y_j$ $(y_1=x, y_j=y)$ is a longest path between x and y in G, $\{x, y\}$ is not a cut set and $j\leqslant n-1$. By Lemma 2.4, $\{y_1, y_j\}\subseteq N_P(R)$ and $|N_P(R)|\geqslant 3$, because $\{x, y\}$ is not a cut set. We consider the following two cases.

Case 1: $x_2^- x_2^+ \notin E$. Choose $\{x_2^-, y_j^-, v\}$ for some $v \in V(R)$ and denote $P_1: y_1 \cdots x_2^-$, $P_2: x_2^+ \cdots y_j$. By Lemma 2.2(ii) and $x_2^- x_2^+ \notin E$, we can get $(N_{P_2}^-(x_2^-) \cup N_{P_2}^-(v)) \cap (N_{P_2}(y_j^-) \cup (N_{P_2}(x_2^-) \cap N_{P_2}(v))) = \emptyset$ and $|N_{P_2}^-(x_2^-) \cup N_{P_2}^-(v)| = |N_{P_2}(x_2^-) \cup N_{P_2}(v)|$. Thus, $\bar{e}(\{x_1^-, y_j^-, v\}; P_2) \leqslant |P_2|$. Therefore, $y_1 \in N_{P_1}^-(y_j^-) \cap N_{P_1}(y_j^-) \cap N_{P_1}(v)$ (otherwise we can get a contradiction as before).

Choose $i^* = \max\{i: y_t \in N_{P_1}(y_j),\ 1 \leqslant t \leqslant i\}$. Clearly, $y_{i^*} \neq x_2$.

If $y_{i^*} x_2^- \notin E$, then $\bar{e}(\{x_2^-, y_j^-, v\};\ P_1) \leqslant |P_1| + 1$, which leads to a contradiction as before.

If $y_{i^*} x_2^- \in E$, then $\{x_2^+, y_{i^*}^+, v\}$ is an independent set of G and $y_i \notin N_{P_1}(y_{i^*}^+) \cup N_{P_1}(x_2^+)$ for $1 \leqslant i \leqslant i^* - 1$. Since $x_2^- x_2^+ \notin E$ and, moreover, $(N_{P_2}^-(x_2^+) \cup N_{P_2}^-(v)) \cap (N_{P_2}(y_{i^*}^+) \cup (N_{P_2}(x_2^+) \cap N_{P_2}(v))) = \phi$, using the same method as before, we can get a contradiction.

Case 2: $x_2^- x_2^+ \in E$. By Lemma 2.3, $x_2^- x_2^{++} \notin E$, $x_2^+ x_2^{--} \notin E$. Choose $\{x_2^{++}, x_1^+, v\}$ for some $v \in V(R)$ and denote $P_1: y_1 \cdots x_2^-$, $P_2: x_2 \cdots y_j$. For the same reason as that in case 2.2 of Lemma 2.4, we can get that $x_2 x_2^{++} \in E$.

Now, we choose $\{x_2^+, y_j^-, v\}$ for some $v \in V(R)$ and denote $P_1: y_1 \cdots x_2$, $P_2: x_2^+ \cdots y_j$. Then, by Lemma 2.2(ii), we have

$$(N_{P_t}^-(x_2^+) \cup N_{P_t}^-(v)) \cap (N_{P_t}(y_j^-) \cup (N_{P_t}(x_2^+) \cap N_{P_t}(v))) = \emptyset \quad (t = 1, 2),$$

and

$$|N_{P_1}^-(x_2^+) \cup N_{P_1}^-(v)| = |N_{P_1}(x_2^+) \cup N_{P_1}(v)| - 1,$$

$$|N_{P_2}^-(x_2^+) \cup N_{P_2}^-(v)| = |N_{P_2}(x_2^+) \cup N_{P_2}(v)|.$$

Thus, by Lemma 2.1, $\bar{e}(\{x_2^+, y_j^-, v\};\ P) \leqslant |P| + 1$, which leads to a contradiction as before.

Therefore, Theorem 1.2 is true.

Acknowledgment

Thanks are due to Prof. H.A. Jung for his helpful suggestions.

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